ISRAEL JOURNAL OF MATHEMATICS **90** (1995), 199–219

SUPER-HIGH-FREQUENCY OSCILLATIONS IN A DISCONTINUOUS DYNAMIC SYSTEM WITH TIME DELAY

BY

EUGENII SHUSTIN

School of Mathematical Sciences
Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel

ABSTRACT

We study oscillations in the discontinuous dynamic system with time delay

$$\dot{x}(t) = -\operatorname{sign} x(t-1) + F(x(t), t), \qquad t \geq 0.$$

This is a typical model of relay feedback with delay. It is known that stable modes in this system have a bounded oscillation frequency. Here we consider transient processes and obtain the following result: under some restrictions on F, the average oscillation frequency of any solution becomes finite after a period of time, i.e. super-high-frequency oscillations (with infinite frequency) exist only in a finite time interval. Moreover, we give an effective upper bound on the length of this interval.

Introduction

It is fruitful for many control problems to use relay control algorithms that yield sliding modes, i.e. a special kind of motion on a surface of discontinuity [AS, F, U]. One of the unavoidable difficulties in realizing such algorithms is the time delay, which is always present in real systems. It results in auto-oscillations [H, KN] and it does not allow one to design an ideal sliding mode. Here we study the system

$$\dot{x}(t) = -\operatorname{sign} x(t-1) + F(x(t), t), \quad t \geq 0, \tag{0.1}$$

where

$$F \in C^2(\mathbb{R}^2), \qquad |F(x,t)| \leq p < 1, \tag{0.2}$$

Received December 20, 1992 and in revised form March 13, 1994

containing simultaneously a discontinuous element and a (relatively) big time delay. This is a model of control when the minimal information is available, that is, only the sign of the output with time delay is known (see, for example, [CH]).

First of all, note that qualitative phenomena in the behavior of system (0.1) cannot be reduced to those for undelayed systems. For example, it is easy to see that any solution of the system

$$\dot{x}(t) = -\operatorname{sign} x(t) + F(x(t), t), \quad t \geq 0,$$
$$F \in C^2(\mathbb{R}^2), \qquad |F(x,t)| \leq p < 1,$$

vanishes identically after a period of time, whereas any solution to (0.1), (0.2) oscillates around the zero level.

Systems with a (relatively) big time delay, of type

$$\dot{x}(t) = f(x(t), x(t-1)),$$

arise in natural sciences, and are studied in many interesting cases [CW, MP, MPN1, MPN2, P, S, W1, W2]. Among their characteristic features we mention: (1) the average oscillation frequency on the delay interval determines mainly the behavior of a solution, (2) there exist infinitely many periodic solutions, or solutions with constant frequency, in the case of discontinuous nonlinearity [MPN1, P], (3) solutions with a non-zero frequency are unstable [MP, W1]. We illustrate this by the following example described in detail in [SFF2, SFF3]; it is a particular case of the system considered in [P]. The simplest equation of type (0.1),

$$\dot{x}(t) = -\operatorname{sign} x(t-1), \tag{0.3}$$

has a countable set of periodic solutions $g_0, g_1, \dots$, with the periods $T_n = 4/(4n+1)$, $n = 0, 1, \dots$:

$$g_0(t) = \begin{cases} t, & -1 \leq t \leq 1, \\ 2-t, & 1 \leq t \leq 3, \end{cases} \qquad g_0(t+4k) = g_0(t), \quad k \in \mathbb{Z},$$

$$g_n(t) = \frac{1}{4n+1} g_0((4n+1)t), \quad t \in \mathbb{R},$$

and any other solution with a finite frequency coincides with one of the above periodic solutions (up to shifts along the t-axis) after a period of time (see [SFF2, SFF3]). In the general case, solutions of (0.1) have similar properties.

On the other hand, the problems of infinite frequency oscillations and of the duration of the transient before finite frequency oscillations are not yet studied. These problems are of interest, because the remote control algorithms related to (0.1) start working only when we have a finite oscillation frequency [SFF1, SFF2]. The aim of the present paper is to treat these problems. Our main result is: under some restrictions on F, any solution of (0.1) has a finite oscillation frequency after some (effectively bounded from above) period of time.

Now let us give precise definitions and statements. It is clear that under condition (0.2), any Cauchy problem

$$x(t) = \varphi(t), \quad t \in [-1;0], \quad \varphi \in C[-1;0],$$

for (0.1) has a unique continuous solution $x_\varphi(t)$, $t \in [-1;\infty)$. The main discrete characteristic of a solution is the following frequency function (see [MP, MPN2]). Let

$$Z_\varphi = \{t \geq -1 \colon x_\varphi(t) = 0\}.$$

Under condition (0.2), Z_φ is, obviously, unbounded. The function

$$\begin{aligned} &\nu_\varphi \colon [0,\infty) \to \mathbb{N} \cup \{0\} \cup \{\infty\}, \\ &\nu_\varphi(t) = \operatorname{card}(Z_\varphi \cap (t^*; t^* - 1)), \quad t^* = \max\{\tau \leq t \colon x_\varphi(\tau) = 0\}, \end{aligned}$$

is called the **frequency** of the solution $x_\varphi(t)$. Similarly to [MP, MPN2], the basic property of this characteristic is described by

LEMMA 0.4: *For any $\varphi \in C[-1;0]$ the function ν_φ does not increase.*

Proof: If $t_1 < t_2$, $t_1, t_2 \in Z_\varphi^+$, then, according to Rolle's Theorem and (0.1), (0.2), there exists $\xi \in (t_1 - 1; t_2 - 1) \cap Z_\varphi$. Therefore

$$\operatorname{card}(Z_\varphi \cap (t_1 - 1; t_2 - 1)) \geq \operatorname{card}\left(Z_\varphi^+ \cap (t_1; t_2)\right) + 1,$$

hence

$$\begin{aligned} \nu_\varphi(t_1) &= \operatorname{card}(Z_\varphi \cap (t_1 - 1; t_1)) \\ &\geq \operatorname{card}(Z_\varphi \cap (t_2 - 1; t_2)) = \nu_\varphi(t_2). \qquad \blacksquare \end{aligned}$$

The meaning of this is that the frequency of any solution becomes constant after a period of time. A priori, this limit frequency may be either finite or infinite. According to Lemma 0.4, in the latter case a solution must have the infinite frequency from the beginning (we call these solutions super-high-frequency steady modes — SHFSM). Our main result is the absence of SHFSM, and an estimate of the super-high-frequency oscillation interval for any solution.

THEOREM 0.5: *There exist positive functions* $d_1, d_2 \in C(-1;1)$ *such that under conditions*

$$F(x,t) = p_0 + xf(x,t), \quad |p_0| < 1, \quad f \in C^1(\mathbb{R}^2), \tag{0.6}$$
$$\max\{|f(x,t)|, |f_x(x,t)|, |f_t(x,t)|\} < d_1(p_0),$$

any solution $x_\varphi(t)$ *with* $\varphi \not\equiv 0$ *has a finite frequency* $\nu_\varphi(t)$ *for*

$$t \geq \frac{d_2(p_0)}{\delta^4} + 1, \tag{0.7}$$

where δ *is the maximal length of a connected component of the set* $[-1;0] - \varphi^{-1}(0)$. *In the case* $\varphi \equiv 0$ *we have:* (i) $x_\varphi \equiv 0$ *when* $p_0 = 0$, (ii) $\nu_\varphi(t) = 0$ *for* $t \geq 1$ *when* $p_0 \neq 0$.

Explicit formulae for d_1, d_2 can be deduced from the proof. Here we give them for the following important particular case:

THEOREM 0.8: *If* $F(x,t) \equiv 0$, *then* $d_2 \equiv 1$. *If* $F(x,t) \equiv p_0 \neq 0$, *then*

$$d_2 = \left(\frac{1+|p_0|}{1-|p_0|}\right)^2 (2\sigma(\lambda,\mu))^{-1},$$

where

$$\lambda = (1+p_0)/2, \quad \mu = (1-p_0)/2,$$
$$\sigma(\lambda,\mu) = \int_0^{1/2} \left(1 - \sqrt{1 - 16\lambda^2\mu^2 \sin^2 2\pi t}\right) dt.$$

Let us add some comments. We note that the study of finite frequency oscillations is reduced to problems on diffeomorphisms of a standard finite-dimensional simplex (see [P], or [SFF2, SFF3] with application to system (0.1)), whereas in the case of infinite frequency we have to deal with operators on the infinite-dimensional symplex. Namely, to a solution we assign an element of the infinite dimensional simplex Σ — the set of lengths of fixed sign intervals on a unit segment, and describe the evolution of this set. It turns out that the shift operator on Σ determined by equation (0.1) is, in a sense, a contraction with respect to a certain norm. Another interpretation of this problem is a random walk on the line, and the question is whether it is ergodic (for example, equation (0.3) generates the random walk with the constant probability 1/2 of moving left or right at each integral point).

Let us add also that conditions (0.6) are, apparently, not necessary, as far as we are interested only in the non-existence of SHFSM. Let us formulate the following

CONJECTURE 0.9: *Let $F(x,t) \equiv F(x)$ and F satisfy (0.2). If $F(0) \neq 0$, then there are no SHFSM. If $F(0) = 0$, then the only one SHFSM is $x(t) \equiv 0$.*

1. Plan of the proof

1.1 MAIN IDEA. We argue by contradiction. First, we introduce some "norm" in the set of closed subsets of a segment, and then we study its evolution for the sequence $Z_\varphi \cap [n; n+1]$, $n \geq 0$. It turns out that, under conditions of Theorem 0.5, for any virtual SHFSM this norm tends to zero as n decreases. On the other hand, in the initial segment this norm is positive. From that we deduce the non-existence of SHFSM and derive an upper bound on the interval with super-high-frequency oscillations.

First we consider the case $F \equiv p_0$. Then we reduce the case $F \not\equiv$ const to the first one by a change of coordinates, which straightens the trajectories of the equations

$$\dot{x} = 1 + F(x,t), \quad \dot{x} = -1 + F(x,t)$$

and transforms the initial equation into

$$\dot{x}(t) = -\operatorname{sign} x(t - 1 - x(t)^2 z(x(t),t)) + p_0. \tag{1.1.1}$$

This equation with a variable delay is treated in the same way as for $F \equiv p_0$.

1.2 THE NORM OF A CLOSED SUBSET OF A SEGMENT AND SHFSM. Let $x_\varphi \not\equiv 0$ be a SHFSM. Let I_φ be the set of isolated points in Z_φ. Put $P_\varphi = Z_\varphi - I_\varphi$. In this situation P_φ is non-empty and unbounded. Let $(\alpha_0; \beta_0)$ be a connected component of $[0; \infty) - P_\varphi$. Then, by (0.2),

$$(\alpha_0 + 1; \beta_0 + 1) \cap P_\varphi = \emptyset$$

and therefore $(\alpha_0 + 1; \beta_0 + 1)$ is contained in some connected component $(\alpha_1; \beta_1)$ of $[0; \infty) - P_\varphi$, and so on. Thus, we obtain a sequence of connected components of the set $[0; \infty) - P_\varphi$:

$$(\alpha_n; \beta_n) \supset (\alpha_{n-1} + 1; \beta_{n-1} + 1), \quad n \geq 1.$$

Lemma 0.4 implies

$$\lim_{n \to \infty} (\beta_n - \alpha_n) = \Delta \leq 1. \tag{1.2.1}$$

Let us denote

$$\alpha = \lim_{n\to\infty} (\alpha_n - n), \qquad \beta = \lim_{n\to\infty} (\beta_n - n).$$

It is easy to see that

$$\alpha_n - k, \beta_n - k \in P_\varphi, \quad n \geq k,$$

hence

$$\alpha + k, \beta + k \in P_\varphi, \quad k \geq 0.$$

Definition 1.2.2: We define $\mathcal{R}$ to be the set of pairs (R, ν), where R is a closed subset of $[\alpha;\beta]$ containing α, β, and ν: $[\alpha,\beta] \to \{0;1;-1\}$ is equal to 0 on R, equal to ± 1 on $[\alpha;\beta] \setminus R$ and is locally constant in $[\alpha;\beta] \setminus R$. By $I(R)$ we denote the set of isolated points in $R \setminus \{\alpha;\beta\}$, and put $P(R) = R \setminus I(R)$. ∎

LEMMA-DEFINITION 1.2.3: *For any $(R,\nu) \in \mathcal{R}$ and $k \geq 1$ there can be found a pair $(R',\nu') \in \mathcal{R}$ such that there exists $x(t) \in C[\alpha;\beta]$ satisfying*

$$\dot{x}(t) = -\nu'(t) + F(x(t), t+k), \quad t \in [\alpha;\beta], \tag{1.2.4}$$

$$\operatorname{sign} x(t) = \nu(t), \ t \in [\alpha;\beta]. \tag{1.2.5}$$

The set of such pairs (R',ν') we denote by $\Pi_k(R,\nu)$.

Proof: We shall construct a shift operator $\mathcal{L}_k$: $\mathcal{R} \to \mathcal{R}$ and show that $(R',\nu') = \mathcal{L}_k(R,\nu)$ is the required pair. Define the function $x(t) \in C[\alpha;\beta]$ by:

(i) $x(t) = 0, \quad t \in R,$

(ii) on each interval $(t_1;t_2) \subset [\alpha;\beta] \setminus R$, $t_1, t_2 \in R$, $\nu(t) = \varepsilon \in \{\pm 1\}$, $t \in (t_1;t_2)$, there exists a solution $x_1(t)$ of the Cauchy problem

$$\dot{x} = \varepsilon + F(x, t+k), \quad x(t_1) = 0,$$

and a solution $x_2(t)$ of the Cauchy problem

$$\dot{x} = -\varepsilon + F(x, t+k), \quad x(t_2) = 0.$$

One of x_1 and x_2 increases and the other decreases. Therefore, there exists a unique $\theta \in (t_1, t_2)$, with $x_1(\theta) = x_2(\theta)$. Put

$$x(t) = \begin{cases} x_1(t), & t_1 \leq t \leq \theta, \\ x_2(t), & \theta \leq t \leq t_2. \end{cases}$$

Evidently, $x(t)$ satisfies (1.2.5). Now put

$$\nu'(t) = \begin{cases} 0, & \text{if } t \in P(R) \text{ or } \dot{x}(t) \text{ does not exist}, \\ F(x(t), t+k) - \dot{x}(t), & \text{otherwise}. \end{cases}$$

Finally, we define $R' = (\nu')^{-1}(0)$. It is easy to verify that (1.2.4) holds as well.

Definition 1.2.6: For any $(R,\nu) \in \mathcal{R}$, define $\|R,\nu\|_\infty$ to be the maximal length of a connected component of $[\alpha;\beta] \setminus R$, and introduce

$$\|R,\nu\| = \sqrt{\mu \sum (l')^2 + \lambda \sum (l'')^2},$$

where l' (resp. l'') runs through the lengths of all connected components of $[\alpha;\beta] \setminus R$ with $\nu(t) > 0$ (resp. $\nu(t) < 0$), $\lambda = (1+p_0)/2, \mu = (1-p_0)/2$. Obviously,

$$\|R,\nu\|_\infty \le \sqrt{\frac{2}{(1+|p_0|)}} \cdot \|R,\nu\|. \tag{1.2.7}$$ ∎

Our goal is to prove

LEMMA 1.2.8: *There exist positive functions $d_1, S_0 \in C(-1;1)$ such that, under conditions* (0.6), *for any sequence*

$$\begin{aligned} &(R,\nu) \in \mathcal{R},\ (R_k,\nu_k) \in \Pi_k(R,\nu),\\ &(R_i,\nu_i) \in \Pi_i(R_{i+1},\nu_{i+1}),\ i = 1,\dots,k-1, \end{aligned} \tag{1.2.9}$$

the inequality

$$\|R_1,\nu_1\|^2 \le \frac{S_0(p_0)}{\sqrt{k+1}} \tag{1.2.10}$$

holds.

Let us deduce Theorem 0.8 from this. Indeed, assume that $x_\varphi(t)$ is a SHFSM. It defines the sequence (1.2.9), where

$$R = [\alpha;\beta] \cap \{t\colon x_\varphi(t+k) = 0\},$$

$$R_m = [\alpha;\beta] \cap \{t\colon x_\varphi(t+m-1) = 0\},\ m = 1,\dots,k,$$

$$\nu(t) = \operatorname{sign} x_\varphi(t+k),\ \nu_m(t) = \operatorname{sign} x_\varphi(t+m-1),\ t \in [\alpha;\beta],\ m = 1,\dots,k.$$

According to Lemma-Definition 1.2.3, this sequence satisfies the conditions of Lemma 1.2.8. Hence, for the set $R_1 = [\alpha;\beta] \cap x_\varphi^{-1}(0)$, we have (1.2.10) with an arbitrary k, and the latter implies that

$$x_\varphi(t) \equiv 0, \quad t \in [\alpha;\beta],$$

which contradicts the initial assumption.

On the other hand, the same argument, (1.2.7) and (1.2.10) yield inequality (0.7) with

$$d_2(p_0) = \left(\frac{2S_0(p_0)}{1+|p_0|}\right)^2.$$

The proof of Lemma 1.2.8 is presented below.

2. Preliminaries

PROPOSITION 2.1: *If positive numbers $a_1, a_2, \dots$ satisfy*

$$a_n \le a_{n-1} - \theta a_{n-1}^3, \quad n \ge 1, \quad \theta = \text{const} > 0, \tag{2.2}$$

then

$$a_n \le \frac{1}{\sqrt{2\theta(n+1)}}.$$

Proof: The inequality (2.2) implies

$$\frac{1}{a_n} \ge \frac{1}{a_{n-1}(1-\theta a_{n-1}^2)} \ge \frac{1}{a_{n-1}}(1+\theta a_{n-1}^2) = \frac{1}{a_{n-1}} + \theta a_{n-1},$$

hence

$$\frac{1}{a_n^2} \ge \frac{1}{a_{n-1}^2} + 2\theta + \theta^2 a_{n-1}^2 \ge \frac{1}{a_0^2} + 2\theta n + \theta^2 \sum_{i=0}^{n-1} a_i^2 \ge 2\theta(n+1). \qquad \blacksquare$$

Thus, instead of (1.2.10), we have to prove

$$\|R', \nu'\|^2 \le \|R, \nu\|^2 - \frac{2}{(1+|p_0|)d_2}\|R, \nu\|^6. \tag{2.3}$$

Definition 2.4: Let $\mathcal{S}$ be the set of sequences $a = \{a_k\}_{k\in\mathbb{Z}}$ of non-negative numbers. Put

$$\|\overline{a}\| = \sqrt{\sum_{k\in\mathbb{Z}} a_k^2}.$$

Let $\lambda > 0,\ \mu > 0,\ \lambda + \mu = 1$. Introduce the linear operator $\overline{b} = L_{\lambda,\mu}(\overline{a})$ in $\mathcal{S}$ by

$$b_{2n} = \lambda a_{2n} + \sqrt{\lambda\mu}\, a_{2n-1}, \quad b_{2n+1} = \mu a_{2n+1} + \sqrt{\lambda\mu}\, a_{2n}, \quad n \in \mathbb{Z}.$$

By L we denote the operator $L_{1/2,1/2}$. $\blacksquare$

PROPOSITION 2.5: *Let*

$$0 < \sum_{n\in\mathbb{Z}} a_n = \gamma < \infty, \quad \overline{b} = L_{\lambda,\mu}(\overline{a}).$$

Then for $\lambda = \mu = 1/2$ we have

$$\|\overline{b}\|^2 \le \|\overline{a}\|^2 - \frac{1}{2\gamma^4}\|\overline{a}\|^6, \tag{2.6}$$

and for $\lambda \neq \mu$ we have

$$\|\bar{b}\|^2 \leq \|\bar{a}\|^2 - \frac{\sigma(\lambda,\mu)}{\gamma^4}\|\bar{a}\|^6, \tag{2.7}$$

where $\sigma(\lambda,\mu)$ is defined in Theorem 0.8.

Proof: Since (2.6), (2.7) are homogeneous, we can put $\gamma = 1$. We start with the proof of (2.6). Consider the functions

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n, \quad g(\tau) = \sum_{n\in\mathbb{Z}} b_n q^n,$$
$$q = \exp(2\pi i\tau), \quad \tau \in [0;1].$$

Put

$$\|f\|^2 = \int_0^1 |f(\tau)|^2 d\tau, \quad \|g\|^2 = \int_0^1 |g(\tau)|^2 d\tau.$$

Then (2.6) is equivalent to

$$\|g\|^2 \leq \|f\|^2 - \frac{1}{2}\|f\|^6.$$

It is easy to show that

$$g(\tau) = f(\tau)\cdot\frac{1+q}{2}.$$

Since $\gamma = f(1) = 1$, then

$$\delta = \|f\|^2 \leq 1, \quad |f(\tau)| \leq 1, \quad \tau \in [0;1],$$

and since $|1+q|^2/4 = \cos^2 \pi\tau$ is an even function that decreases in $[0;1/2]$, we can apply Steffensen's inequality (see [BB, Theorem 32]), which says that

$$\int_a^b m(t)n(t)dt \leq \int_a^{a+c} n(t)dt, \quad c = \int_a^b m(t)dt, \tag{2.8}$$

where $0 \leq m(t) \leq 1$, $n(t) \geq 0$, $n'(t) \leq 0$, $t \in [a;b]$. Thus we obtain

$$\|g\|^2 = \int_{-1/2}^{1/2} |f(\tau)|^2 \cos^2 \pi\tau d\tau \leq 2\int_0^{\delta/2} \cos^2 \pi\tau d\tau$$
$$= \delta - \delta^3 \frac{\delta - (\sin\pi\delta)/\pi}{2\delta^3} \leq \delta - \frac{1}{2}\delta^3.$$

Now we prove (2.7). To any $\bar{a} \in \mathcal{S}$ we assign the vector-function

$$\bar{f}(\tau) = (f_0(\tau), \quad f_1(\tau)),$$

$$f_0(\tau)=\sum_{k\in\mathbb{Z}} a_{2k}q^{2k},\quad f_1(\tau)=\sum_{k\in\mathbb{Z}} a_{2k+1}q^{2k+1},$$
$$q=\exp(2\pi i\tau),\quad \tau\in[0;1].$$

Put

$$\|\overline{f}\|^2=\int_0^1(|f_0(\tau)|^2+|f_1(\tau|^2)d\tau.$$

Let $\overline{g}(\tau)=(g_0(\tau),g_1(\tau))$ be assigned to $\overline{b}=L_{\lambda,\mu}(\overline{a})$. Then (2.7) is equivalent to

$$\|\overline{g}\|^2\le\|\overline{f}\|^2-\sigma(\lambda,\mu)\cdot\|\overline{f}\|^6.$$

It is easy to see that $\overline{g}$ can be obtained from $\overline{f}$ by multiplication by the matrix

$$M=\begin{pmatrix}\lambda & \sqrt{\lambda\mu}q\\ \sqrt{\lambda\mu}q & \mu\end{pmatrix}.$$

Now we observe that the maximal eigenvalue of the matrix $\overline{M}M$ is equal to

$$\xi(\tau)=\frac{1+\sqrt{1-16\lambda^2\mu^2\sin^2 2\pi\tau}}{2}.$$

Therefore

$$|g_0(\tau)|^2+|g_1(\tau)|^2\le\xi(\tau)(|f_0(\tau)|^2+|f_1(\tau)|^2),$$

$$\|\overline{g}\|^2\le 2\int_0^{1/4}\xi(\tau)(|f_0(\tau)|^2+|f_1(\tau)|^2)d\tau. \tag{2.9}$$

Since $\gamma=f_0(1)+f_1(1)=1$, then

$$\delta=\|\overline{f}\|^2\le 1,\quad |f_0(\tau)|^2+|f_1(\tau)|^2\le 1,\quad \tau\in[0;1]. \tag{2.10}$$

Obviously, $\xi(\tau)$ decreases in $[0;1/4]$. Hence due to Steffensen's inequality (2.8) and (2.9), (2.10) we get

$$\|\overline{g}\|^2\le 4\int_0^{\delta/4}\xi(\tau)d\tau=\delta-2\int_0^{\delta/4}\left(1-\sqrt{1-16\lambda^2\mu^2\sin^2 2\pi\tau}\right)d\tau. \tag{2.11}$$

Let us show that the function

$$2\delta^{-3}\int_0^{\delta/4}\left(1-\sqrt{1-16\lambda^2\mu^2\sin^2 2\pi\tau}\right)d\tau$$

is decreasing in δ. Indeed, this follows from

$$\left(2\int_0^{\delta/4}\left(1-\sqrt{1-16\lambda^2\mu^2\sin^2 2\pi\tau}\right)d\tau\right)' = \frac{1}{2}\left(1-\sqrt{1-16\lambda^2\mu^2\sin^2\frac{\pi\delta}{2}}\right)$$
$$\leq \frac{1}{2}\left(1-\sqrt{1-\sin^2\frac{\pi\delta}{2}}\right) = \sin^2\frac{\pi\delta}{4} \leq 3\delta^2 = (\delta^3)'.$$

Hence

$$2\delta^{-3}\int_0^{\delta/4}\left(1-\sqrt{1-16\lambda^2\mu^2\sin^2 2\pi\tau}\right)d\tau$$
$$\leq 2\int_0^{1/4}\left(1-\sqrt{1-16\lambda^2\mu^2\sin^2 2\pi\tau}\right)d\tau = \sigma(\lambda,\mu),$$

and hence (2.11) yields

$$\|\bar{g}\| \leq \delta - \sigma(\lambda,\mu)\delta^3.$$

PROPOSITION 2.12: *Let $\{\delta_k\}, k\in\mathcal{K}, \{\gamma_k\}, k\in\mathcal{K}$, be sets of positive numbers, and*

$$\sum\delta_k = \delta < \infty, \quad \sum\gamma_k = \gamma < \infty.$$

Then

$$\sum_{k\in\mathcal{K}}\frac{\delta_k^3}{\gamma_k^4} \geq \frac{\delta^3}{\gamma^4}.$$

Proof: Due to homogeneity we can assume $\delta = \gamma = 1$, hence $\delta_k \geq \gamma_k$ for at least one k, which implies the required inequality.

PROPOSITION 2.13: *Under the conditions of Proposition 2.5, we have for $\lambda = \mu = 1/2$*

$$\sum_{n\in\mathbb{Z}}(a_n - a_{n-1})^2 = 4(\|\bar{a}\|^2 - \|\bar{b}\|^2), \tag{2.14}$$

$$\sum_{n\in\mathbb{Z}} a_n^{s+2} \leq 4\gamma^s(\|\bar{a}\|^2 - \|\bar{b}\|^2), \quad s\geq 2. \tag{2.15}$$

Remark 2.16: Proof of inequality (2.15) for $s = 2$ was suggested to me by Prof. V. Matsaev. I would like to express my gratitude to V. Matsaev for his help and kind permission to present his proof here. ∎

Proof of Proposition 2.13: Formula (2.14) is trivial. Inequalities (2.15) with $s > 2$ follow from the same inequality with $s = 2$. Now consider the following simple inequality:

$$|a_n^{3/2} - a_{n-1}^{3/2}| \leq |a_n - a_{n-1}|(\sqrt{a_n} + \sqrt{a_{n-1}}).$$

This implies

$$2\max_{n\in\mathbb{Z}} a_n^{3/2} \leq \sum_{n\in\mathbb{Z}} |a_n^{3/2} - a_{n-1}^{3/2}| \leq \sum_{n\in\mathbb{Z}} |a_n - a_{n-1}|(\sqrt{a_n} + \sqrt{a_{n-1}})$$
$$\leq \sqrt{\sum_{n\in\mathbb{Z}}(a_n - a_{n-1})^2 \cdot \sum_{n\in\mathbb{Z}}(\sqrt{a_n} + \sqrt{a_{n-1}})^2} \leq 2\sqrt{\sum_{n\in\mathbb{Z}}(a_n - a_{n-1})^2 \cdot \sum_{n\in\mathbb{Z}} a_n},$$

and, finally,

$$\sum_{n\in\mathbb{Z}} a_n^4 \leq \max_{n\in\mathbb{Z}} a_n^3 \cdot \sum_{n\in\mathbb{Z}} a_n \leq \sum_{n\in\mathbb{Z}}(a_n - a_{n-1})^2 \cdot (\sum_{n\in\mathbb{Z}} a_n)^2,$$

which is equivalent to (2.15) for $s = 2$.

PROPOSITION 2.17: *Under the conditions of Proposition 2.5, we have for* $\lambda < \mu$

$$\sum_{n\in\mathbb{Z}} \Big(\lambda(\sqrt{\mu}a_{2n} - \sqrt{\lambda}a_{2n-1})^2 + \mu(\sqrt{\lambda}a_{2n+1} - \sqrt{\mu}a_{2n})^2\Big) = \|\bar{a}\|^2 - \|\bar{b}\|^2, \tag{2.18}$$

$$\sum_{n\in\mathbb{Z}} a_n^{s+2} \leq \frac{\mu^{s/2}\gamma^s}{\lambda^{s/2+2}}(\|\bar{a}\|^2 - \|\bar{b}\|^2), \quad s \geq 2. \tag{2.19}$$

Proof: Formula (2.18) can be derived by an elementary computation. Inequality (2.19) is a consequence of (2.18) and (2.15).

3. The case $F \equiv p_0$

Consider the equation

$$\dot{x}(t) = -\operatorname{sign} x(t-1) + p_0, \quad p_0 \in (-1, 1). \tag{3.1}$$

It is autonomous, hence the operators $\mathcal{L}_k$ and the sets $\Pi_k(R,\nu)$ do not depend on k. So, we write $\mathcal{L}$ instead of $\mathcal{L}_k$, and $\Pi(R,\nu)$ instead of $\Pi_k(R,\nu)$ in this section.

As the first step we state

PROPOSITION 3.2: *Any* $(R,\nu)\in\mathcal{R}$, $(R',\nu')\in\Pi(R,\nu)$ *satisfy*

$$\|\mathcal{L}(R,\nu)\|\geq\|R',\nu'\|.$$

Proof: Let (R',ν') belong to $\Pi(R,\nu)$, and let the function $x(t)\in C[\alpha;\beta]$ satisfy

$$\dot{x}(t)=-\nu'(t)+p_0,\quad \operatorname{sign}x(t)=\nu(t),\quad t\in[\alpha;\beta].$$

For any component $(t';t'')$ of the set $[\alpha;\beta]\setminus R'$ define $\sigma(t',t'')$ to be the rectangle in the plane (t,x) formed by the lines

$$t=t',\ t=t'',\ x=x(t'),\ x=x(t'').$$

Evidently, $\sigma(t',t'')$ contains a linear segment of the graph of $x(t)$ as a diagonal. Denote

$$\sigma(R',\nu')=\bigcup_{(t';t'')}\sigma(t',t''),$$

where $(t';t'')$ runs through all components of the set $[\alpha;\beta]-R'$. It is easy to see that the area of $\sigma(t',t'')$ is equal to

$$S(\sigma(t',t''))=\begin{cases}\lambda(t''-t')^2,\ \nu'\big|_{t';t'')}=-1,\\ \mu(t''-t')^2,\ \nu'\big|_{(t';t'')}=1,\end{cases}$$

hence $\|R',\nu'\|^2$ is equal to the area of $\sigma(R',\nu')$.

Let us show that

$$\sigma(R',\nu')\subset\sigma(\mathcal{L}(R,\nu)). \tag{3.3}$$

Fix a component $(t_1;t_2)$ of the set $[\alpha;\beta]\setminus R$. By definition, the graph of the function $y(t)$ satisfying

$$\dot{y}(t)=-\mathcal{L}(\nu)(t)+p_0,\quad \operatorname{sign}y(t)=\nu(t),\quad t\in(t_1;t_2),$$

consists of two segments. It is easy to see that the graph of $x(t)$ on $(t_1;t_2)$ is contained in the triangle bounded by the t-axis and by the graph of $y(t)$. This immediately implies (3.3), and thereby completes the proof. ∎

For an element $(R,\nu)\in\mathcal{R}$, let us define

$$\|R,\nu\|_1=\sqrt{\mu}\sum{}' a_k+\sqrt{\lambda}\sum{}'' a_k,$$

where $\sum'$ (resp. $\sum''$) is taken over the lengths of all connected components of $[\alpha;\beta]-R$ with $\nu=1$ (resp. $\nu=-1$).

PROPOSITION 3.4: *If $p_0 = 0$ then*

$$\|\mathcal{L}(R,\nu)\|^2 \le \|R,\nu\|^2 - \frac{1}{2\|R,\nu\|_1^4}\|R,\nu\|^6. \tag{3.5}$$

If $p_0 \neq 0$ then

$$\|\mathcal{L}(R,\nu)\|^2 \le \|R,\nu\|^2 - \frac{\sigma(\lambda,\mu)}{\|R,\nu\|_1^4}\|R,\nu\|^6. \tag{3.6}$$

Proof:

STEP 1: Assume that $p_0 = 0$ and R is locally finite in $(\alpha;\beta)$. Then, starting with any component of the set $(\alpha;\beta)\setminus R$, we can successively renumber the lengths $\{a_k\}_{k\in\mathbb{Z}}$ of all components of $(\alpha;\beta)\setminus R$ in the following way. Let $(t;t'),(t';t'')$ be neighbouring components of $(\alpha;\beta) - R$, and $t'-t = a_s$. If ν changes its sign at t' then put $t''-t' = a_{s+1}$, if not then put $a_{s+1} = 0$, $t''-t' = a_{s+2}$. Now we observe that the action of $\mathcal{L}$ on (R,ν) corresponds to the action of the operator L on $\{a_k\}_{k\in\mathbb{Z}}$, and

$$\|R,\nu\| = \frac{\|\overline{a}\|}{\sqrt{2}}, \quad \|\mathcal{L}(R,\nu)\| = \frac{\|L(\overline{a})\|}{\sqrt{2}}, \quad \|R,\nu\|_1 = \frac{\sum a_k}{\sqrt{2}},$$

hence (3.5) follows from (2.6).

STEP 2: Assume that $p_0 = 0$, and R is not locally finite in $(\alpha;\beta)$. For any component $(t_1;t_2)$ of the set $(\alpha;\beta)\setminus P(R)$ (see Definition 1.2.2) the intersection R' of R with $(t_1;t_2)$ is locally finite, hence by Step 1

$$\|\mathcal{L}(R',\nu)\|^2 \le \|R',\nu\|^2 - \frac{1}{2\|R',\nu\|_1^4}\|R',\nu\|^6.$$

Summing up over all components of the set $(\alpha;\beta)\setminus P(R)$, we derive, by Proposition 2.12:

$$\begin{aligned}\|\mathcal{L}(R,\nu)\|^2 &= \sum \|\mathcal{L}(R',\nu)\|^2 \le \sum \|R',\nu\|^2 - \sum \frac{\|R',\nu\|^2}{2\|R',\nu\|_1^4} \\ &\le \sum \|R',\nu\|^2 - \frac{(\sum \|r',\nu\|^2)^3}{2(\sum \|R',\nu\|_1)^4} = \|R,\nu\|^2 - \frac{\|R,\nu\|^6}{2\|R,\nu\|_1^4}.\end{aligned}$$

STEP 3: Assume that $p_0 \neq 0$. If R is locally finite in $(\alpha;\beta)$, then we renumber the lengths $\{a_k\}$ of all components of the complement of R in $(\alpha;\beta)$ as described in the first step, and so that components with positive ν have even numbers. Then we define the sequence $\overline{a}' = \{a'_k\} \in \mathcal{S}$ by

$$a'_{2k} = \sqrt{\mu} a_{2k}, \quad a'_{2k+1} = \sqrt{\lambda} a_{2k+1}, \quad k \in \mathbb{Z},$$

and, in a similar way, show that (3.6) is equivalent to inequality (2.7) applied to $\overline{a}'$. Finally, the case of a general set R is reduced to the locally finite case as it was done in the second step. ∎

Finally, we see that

$$\|R, \nu\|_1 \geq \frac{1 - |p_0|}{2} \|R, \nu\|_\infty,$$

and hence (3.5), (3.6) imply (2.3) with d_2 given by Theorem 0.8.

4. The case $F \not\equiv$ const

Consider the equation

$$\dot{x}(t) = -\operatorname{sign} x(t-1) + p_0 + x(t) f(x(t), t) \tag{4.1}$$

where

$$f(x,t) \in C^1(\mathbb{R}^2), \quad |p_0 + x f(x,t)| \leq p < 1. \tag{4.2}$$

PROPOSITION 4.3: *Under condition* (4.2) *there exists a coordinate change in the plane* $\mathcal{T}(t,x) = (\tau, \theta)$ *of the type*

$$\begin{cases} \theta = x + x^2 \Phi(x,t), \\ \tau = t + x^2 \Psi(x,t), \end{cases} \tag{4.4}$$

that transforms the trajectories of the equations

$$\dot{x} = -1 + p_0 + x f(x,t), \quad \dot{x} = 1 + p_0 + x f(x,t), \tag{4.5}$$

into straight lines

$$d\theta = (-1 + p_0) d\tau, \quad d\theta = (1 + p_0) d\tau.$$

Proof: Put

$$(1 + p_0)\Psi - \Phi = \xi, \quad (-1 + p_0)\Psi - \Phi = \eta. \tag{4.6}$$

Then the conditions on $\mathcal{T}$ can be written down as

$$(4.7)\qquad \begin{aligned} \xi &= \frac{(f + x(-2\xi f - \xi_t) + (1+p_0)\xi_x + xf\xi_x}{2+2p_0}, \\ \eta &= \frac{f + x(-2\eta f - \eta_t) + (-1+p_0)\eta_x + xf\eta_x}{-2+2p_0}. \end{aligned}$$

The characteristic lines of these equations are just the trajectories of equations (4.5), and hence, according to (4.2), equations (4.7) are solvable in the plane with initial data

$$(4.8)\qquad \xi\big|_{x=0} = \frac{f(0,t)}{2+2p_0}, \quad \eta\big|_{x=0} = \frac{f(0,t)}{-2+2p_0}.$$

This completes the proof. ∎

From (4.4) we immediately obtain the formulae for $\mathcal{T}^{-1}$:

$$(4.9)\qquad \begin{cases} x = \theta + \theta^2 s(\theta,\tau) \\ t = \tau + \theta^2 r(\theta,\tau) \end{cases} \quad s, r \in C^1(\mathbb{R}^2).$$

Let $x(t)$ be a solution of (4.1). The diffeomorphism $\mathcal{T}$ takes $x(t)$ to a piecewise linear function $\theta(\tau)$ with

$$\dot\theta(\tau) = \begin{cases} 1+p_0, & \dot x(t) > 0, \\ -1+p_0, & \dot x(t) < 0. \end{cases}$$

Now we observe that, according to (4.4), (4.7) and (4.8), the following holds on the line $\theta = 0$:

$$x = 0, \quad t = \tau, \quad \frac{D(x,t)}{D(\theta,\tau)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In particular, this means that $x(t)$ and $\theta(\tau)$ have the same set of zeroes, and

$$\operatorname{sign}\theta(\tau) = \operatorname{sign} x(t).$$

Combining all these observations, we obtain that $\theta(\tau)$ satisfies the equation

$$(4.10)\qquad \dot\theta(\tau) = -\operatorname{sign}\theta(\tau - 1 + \theta(\tau)^2 r(\theta(\tau),\tau)) + p_0.$$

Now we can describe the shift operator $\mathcal{L}_k$ for equation (4.1) via the shift operator of equation (4.10). Namely, for any $(R,\nu) \in \mathcal{R}$ we can construct, as described in Lemma 1.2.3, the set $\Pi_k(R,\nu)$ and the element $\mathcal{L}_k(R,\nu) \in \Pi_k(R,\nu)$ for equation (4.1), and the set $\Pi(R,\nu)$ and the element $\mathcal{L}(R,\nu) \in \Pi(R,\nu)$ for equation (3.1). Due to the equivalence of (4.1) and (4.10), we have the bijective mapping

$$M\colon \Pi(R,\nu) \longrightarrow \Pi_k(R,\nu),$$

which acts as follows. To an element $(R',\nu') \in \Pi(R,\nu)$ we assign the function $x(t)$ satisfying

$$\dot{x}(t) = -\nu'(t) + p_0,$$

$$\operatorname{sign} x(t) = \nu(t), \quad t \in [\alpha;\beta],$$

and then we set $M(R',\nu')$ to be equal to the image of (R',ν') under the homeomorphism

$$t \in [\alpha;\beta] \mapsto \tau = t + x^2(t)\Psi(x(t), t+k) \in [\alpha;\beta].$$

Obviously,

$$M(\mathcal{L}(R,\nu)) = \mathcal{L}_k(R,\nu). \tag{4.11}$$

PROPOSITION 4.12: *For any $(R'\nu') \in \Pi_k(R,\nu)$,*

$$\|\mathcal{L}_k(R,\nu)\| \geq \|R',\nu'\|.$$

The proof coincides with that of Proposition 3.2.

Assume that

$$\sup_t |\Psi(0;t)| \leq \delta, \quad \sup_t |\Psi_t(0;t)| \leq \delta_t, \quad \sup_{\substack{t\in\mathbb{R} \\ |x|\leq(1-p_0^2)/2}} |\Psi_x(x,t)| \leq \delta_x, \tag{4.13}$$

where δ, δ_t, $\delta_x \in \mathbb{R}$. Denote by $\mathcal{W}$ the set of polynomials

$$\sum_{i_1+i_2+i_3>0} \delta^{i_1}\delta_x^{i_2}\delta_t^{i_3}\xi_{i_1 i_2 i_3}(\lambda,\mu),$$

where $\xi_{i_1 i_2 i_3}(\lambda,\mu)$ are rational functions of λ,μ with only positive coefficients.

PROPOSITION 4.14: *There exists a polynomial $w \in \mathcal{W}$ with the property: if*

$$w_0 = w(\delta,\delta_x,\delta_t,\lambda,\mu) < 1,$$

then

(1) *for* $p_0 = 0,\ k \geq 1,\ (R,\nu) \in \mathcal{R}$

$$\|\mathcal{L}_k(R,\nu)\|^2 \leq \|R,\nu\|^2 - \frac{1-w_0}{2\|R,\nu\|_1^4}\|R,\nu\|^6 \ ;$$

(2) *for* $p_0 > 0, k \geq 1, (R,\nu) \in \mathcal{R}$

$$\|\mathcal{L}_k(R,\nu)\|^2 \leq \|R,\nu\|^2 - \frac{(1-w_0)\sigma(\lambda,\mu)}{\|R,\nu\|_1^4}\|R,\nu\|^6.$$

Proof: As explained in the proof of Proposition 3.4, according to Proposition 2.12, it is sufficient to study the case of a locally finite set $R \cap (\alpha;\beta)$. Let $\{a_n\}_{n\in\mathbb{Z}}$ be the lengths of naturally ordered connected components of $[\alpha;\beta] \setminus R$. Then the lengths $\{b_n\}_{n\in\mathbb{Z}}$ of connected components of $[\alpha;\beta] \setminus \mathcal{L}_k(R)$ can be expressed by means of (4.11) as

$$\begin{aligned} b_{2n} =& \lambda(a_{2n}+a_{2n-1}) + 4\lambda^2\mu^2a_{2n}^2\Psi(2\lambda\mu a_{2n}, t_{2n}+k) \\ &- 4\lambda^2\mu^2a_{2n-1}^2\Psi(-2\lambda\mu a_{2n-1}, t_{2n-1}+k), \\ b_{2n+1} =& \mu(a_{2n+1}+a_{2n}) + 4\lambda^2\mu^2a_{2n+1}^2\Psi(-2\lambda\mu a_{2n+1}, t_{2n+1}+k) \\ &- 4\lambda^2\mu^2a_{2n}^2\Psi(2\lambda\mu a_{2n}, t_{2n}+k), \\ t_{2n} =& \sum_{i<2n} a_i + \lambda a_{2n}, \quad t_{2n+1} = \sum_{i<2n+1} a_i + \mu a_{2n+1}, \quad n \in \mathbb{Z}. \end{aligned}$$

In what follows we use sequences $\{a'_n\}, \{b'_n\}$:

$$\begin{aligned} a'_{2n} &= a_{2n}\sqrt{\mu}, \quad a'_{2n+1} = a_{2n+1}\sqrt{\lambda}, \\ b'_{2n} &= b_{2n}\sqrt{\mu}, \quad b'_{2n+1} = b_{2n+1}\sqrt{\lambda}, \quad n \in \mathbb{Z}. \end{aligned}$$

Let us introduce the notation $\overline{b}^{(0)} = L_{\lambda,\mu}(\overline{a}')$, so that

$$b_{2n}^{(0)} = \lambda a'_{2n} + \sqrt{\lambda\mu}a'_{2n-1}, \ b_{2n+1}^{(0)} = \sqrt{\lambda\mu}a'_{2n} + \mu a'_{2n+1};$$

we thus obtain

$$\begin{aligned} b'_{2n} &= b_{2n}^{(0)} + 4\lambda^2\mu^{3/2}a_{2n}^{'2}\Psi(2\lambda\sqrt{\mu}a'_{2n}, t_{2n}+k) \\ &\quad - 4\lambda\mu a^{5/2}a_{2n-1}^{'2}\Psi(-2\sqrt{\lambda}\mu a'_{2n-1},\ t_{2n-1}+k), \\ b'_{2n+1} &= b_{2n+1}^{(0)} + 4\lambda^{3/2}\mu^2a_{2n+1}^{'2}\Psi(-2\sqrt{\lambda}\mu a'_{2n+1}, t_{2n+1}+k) \\ &\quad - 4\lambda^{5/2}\mu a_{2n}^{'2}\Psi(2\lambda\sqrt{\mu}a'_{2n},\ t_{2n}+k), \end{aligned}$$

and hence, by (4.13),

$$\begin{aligned} b'_{2n} \le b^{(0)}_{2n} &+ 4\lambda\mu^{3/2}(\lambda a'^2_{2n} - \mu a'^2_{2n-1})\Psi(0; t'_{2n}+k) \\ &+ 4\lambda^3\mu a'^3_{2n}(2\mu\delta_1+\delta_2) + 4\lambda^{1/2}\mu^{7/2}a'^3_{2n-1}(2\lambda\delta_1+\delta_2), \\ b'_{2n+1} \le b^{(0)}_{2n+1} &+ 4\lambda^{3/2}\mu(\mu a'^2_{2n-1} - \lambda a'^2_{2n})\Psi(0; t'_{2n+1}+k) \\ &+ 4\lambda\mu^3 a'^2_{2n+1}(2\lambda\delta_1+\delta_2) + 4\lambda^{7/2}\mu^{1/2}a'^2_{2u}(2\mu\delta_1+\delta_2). \end{aligned}$$

where

$$t'_l = \sum_{i<l} a_i, \quad l \in \mathbb{Z}.$$

Now elementary computations yield

$$\|\bar{b}'\|^2 \le \|\bar{b}^{(0)}\|^2 + w_4\cdot\sum_{n\in\mathbb{Z}} a'^4_n + w_5\cdot\sum_{n\in\mathbb{Z}} a'^5_n + w_6\cdot\sum_{n\in\mathbb{Z}} a'^6_n + 8\lambda^{3/2}\mu^{3/2}\sum_{n\in\mathbb{Z}} A_n, \tag{4.15}$$

where $w_4, w_5, w_6 \in \mathcal{W}$ and

$$\begin{aligned} A_n =&(\sqrt{\lambda}a'_{2n} + \sqrt{\mu}a'_{2n-1})(\lambda a'^2_{2n} - \mu a'^2_{2n-1})\Psi(0, t'_{2n}+k) \\ &+ (\sqrt{\mu}a'_{2n+1} + \sqrt{\lambda}a'_{2n})(\mu a'^2_{2n+1} - \lambda a'^2_{2n})\Psi(0, t'_{2n+1}+k)). \end{aligned}$$

Each term in the latter formula can be transformed as follows:

$$\begin{aligned} (\sqrt{\lambda}a'_{2n} + \sqrt{\mu}a'_{2n-1})(\lambda a'^2_{2n} - \mu a'^2_{2n-1}) = \frac{3\lambda-\mu}{3\sqrt{\lambda}}a'^3_{2n} - \frac{3\mu-\lambda}{3\sqrt{\mu}}a'^3_{2n-1} \\ + \left(\frac{\lambda^2-3\mu+\lambda}{3\sqrt{\lambda}\mu}a'_{2n} - \frac{\mu^2-3\lambda+\mu}{3\lambda\sqrt{\mu}}a'_{2n-1}\right)\left(\sqrt{\mu}a'_{2n} - \sqrt{\lambda}a'_{2n-1}\right)^2. \end{aligned}$$

Therefore, the last sum in (4.15) can be bounded from above by

$$\begin{aligned} &8\lambda^{3/2}\mu^{3/2}\delta\rho(\lambda,\mu)\|R,\nu\|_1 \sum_{n\in\mathbb{Z}}\Bigg((\sqrt{\mu}a'_{2n} - \sqrt{\lambda}a'_{2n-1})^2 \\ &\qquad\qquad + \mu(\sqrt{\lambda}a'_{2n+1} - \sqrt{\mu}a'_{2n})^2\Bigg) \\ &\qquad + 8\lambda^{3/2}\mu^{3/2}\delta_t \sum_{n\in\mathbb{Z}}\left(\frac{|3\lambda-\mu|}{3\sqrt{\lambda}}a'^4_{2n} + \frac{|3\mu-\lambda|}{3\sqrt{\mu}}a'^4_{2n+1}\right), \end{aligned} \tag{4.16}$$

where $\rho(\lambda,\mu)$ is a rational function of λ, μ. Finally, combining (4.15), (4.16), (2.18) and (2.19), we obtain

$$\|\bar{b}'\|^2 \le \|\bar{b}^{(0)}\|^2 + \widetilde{w}\cdot(\|\bar{a}'\|^2 - \|\bar{b}^{(0)}\|^2), \quad \widetilde{w} \in \mathcal{W},$$

hence, by (2.6), (2.7),

$$\|\overline{b}'\|^2 \le \|\overline{a}'\|^2 - (1-\widetilde{w})(\|\overline{a}'\|^2 - \|\overline{b}^{(0)}\|^2)$$
$$\le \|\overline{a}'\|^2 - \begin{cases} (1-\widetilde{w})\|\overline{a}'\|^6/(2\|\overline{a}'\|_1^4), & \text{if } p_0 = 0, \\ (1-\widetilde{w})\sigma(\lambda,\mu)\|\overline{a}'\|^6/\|\overline{a}'\|_1^4, & \text{if } p_0 > 0. \end{cases} \quad ∎$$

The final step in the proof of Theorem 0.5 is

PROPOSITION 4.17: *Let $w \in \mathcal{W}$ be a polynomial of Proposition 4.14. For any continuous positive functions $\delta = \delta(p_0)$, $\delta_x = \delta_x(p_0)$, $\delta_t = \delta_t(p_0)$ satisfying inequality $w(\delta, \delta_x, \delta_t, \lambda, \mu) < 1$, there exists a continuous positive function $d_1(p_0)$ with the property: if*

$$\max\{|f(x,t)|,\ |f_x(x,t)|,\ |f_t(x,t)|\} \le d_1(p_0),$$

then $\Psi(x,t)$ satisfies (4.13).

Proof: The statement can be easily deduced from (4.6), (4.7), (4.8).

ACKNOWLEDGEMENT: I would like to thank the referee for many helpful remarks and suggestions, which allowed me to improve the text.

References

[AS] J. Andre and P. Seifert, *Über die Stückweise lineare Differenzialgleichungen, die bei Regelungsproblem auftreten,* Archiv für Mathematik **7**, (1956), no. 3, 148–164.

[BB] E.F. Beckenbach E.F and R. Bellman, *Inequalities (Ergebnisse der Mathematik und ihrer Grenzgebiete, H. 30),* Springer, Berlin, 1961.

[CH] D. Cho and K.J. Hedrick, *Sliding mode fuel-injection controller: its advantages,* ASME Journal of Dynamic Systems, Measurement and Control (1994), to appear.

[CW] S. N. Chow and H-O. Walther, *Characteristic multipliers and stability of symmetric periodic solutions of $\dot{x}(y) = d(x(t-1))$,* Transactions of the American Mathematical Society **307** (1988) no. 1, 127–142.

[F] A.F. Filippov, *Differential Equations with the Discontinuous Right Hand Side,* Kluwer Academic Press, Dordrecht,, 1988.

[H] J. Hale, *Theory of Functional Differential Equations,* Springer, Berlin, 1979.

[KN] V.B. Kolmanovsky and V.R. Nosov, *Stability and Periodic Modes of Controllable Systems with Time Delay,* Nauka, Moscow, 1981 (Russian).

[MP] J. Mallet-Paret, *Morse decomposition for delay-differential equations,* Journal of Differential Equations **72** (1988), 270–315..

[MPN1] J. Mallet-Paret and R.D. Nussbaum, *A bifurcation gap for a singularly perturbed delay equation,* in *Chaotic Dynamics and Fractals,* Vol. 2 (M.F. Barnsley and S.G. Demko, eds.), Acadamic Press, New York, 1986, pp. 263–287.

[MPN2] J. Mallet-Paret and R.D. Nussbaum, *Boundary layer phenomena for differential-delay equations with state-dependent time lags, I.,* Archives for Rational Mechanics and Analysis **120** (1992), 99–146.

[P] H. Peters, *Comportement chaotique d'une équation differentielle retardée,* Comptes Rendus de l'Academie de Sciences, Paris **290** (1980), 1119–1122.

[S] D. Saupe, *Global bifurcation of periodic solutions to some autonomous differential delay equation,* Applied Mathematics and Computation **13** (1983), 185–211.

[SFF1] E. Shustin, E. Fridman and L. Fridman, *Adaptive delay control of relay type in autonomous and periodic systems,* in *Proceedings of the Second European Control Conference, Groningen, July 1993,* Vol. 4, pp. 2212–2215.

[SFF2] E.I. Shustin, E.M. Fridman and L.M. Fridman, *Relay control with time delay in autonomous and periodic systems,* Preprint, Tel Aviv University, 1992 (to appear in: Proc. Internat. Conf. "Journnees non-holonomes", Paris, July 1–3, 1992).

[SFF3] E. Shustin, E. Fridman and L. Fridman, *Steady modes in a discontinuous control system with time delay,* Pure Mathematics and Applications **4** (1993), no. 1, 59–73.

[U] V.I. Utkin, *Sliding Modes in Control Optimization,* Springer, Berlin, 1991.

[W1] H.-O. Walther, *Density of slowly oscillating solutions of $\dot{x}(t) = -f(x(t-1))$,* Journal of Mathematical Analysis and its Applications **79** (1981), 127–140.

[W2] H.-O. Walther, *An invariant manifold of slowly oscillating solutions for $\dot{x}(t) = -\mu x(t) + f(x(t-1))$,* Journal für die Reine Angewandte Mathematik **414** (1991), 67–112.